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Hierarchical convergence of an implicit double-net algorithm for nonexpansive semigroups and variational inequality problems

Yonghong Yao[1], Yeol Je Cho[2*] and Yeong-Cheng Liou[3]

* Correspondence: yjcho@gsnu.ac.kr
[2]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we show the hierarchical convergence of the following implicit double-net algorithm:

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)(x_{s,t} - \mu Ax_{s,t})] + (1-s)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(v)x_{s,t}dv, \quad \forall s, t \in (0,1),$$

where f is a ρ-contraction on a real Hilbert space H, $A : H \to H$ is an α-inverse strongly monotone mapping and $S = \{T(s)\}_{s \geq 0}$: $H \to H$ is a nonexpansive semigroup with the common fixed points set $Fix(S) \neq \emptyset$, where $Fix(S)$ denotes the set of fixed points of the mapping S, and, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm as $s \to 0$ to a common fixed point $x_t \in Fix(S)$ of $\{T(s)\}_{s \geq 0}$and, as $t \to 0$, the net $\{x_t\}$ converges in norm to the solution x^* of the following variational inequality:

$$\begin{cases} x^* \in Fix(S); \\ \langle Ax^*, x - x^* \rangle \geq 0, \forall x \in Fix(S). \end{cases}$$

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1 Introduction

In nonlinear analysis, a common approach to solving a problem with multiple solutions is to replace it by a family of perturbed problems admitting a unique solution and to obtain a particular solution as the limit of these perturbed solutions when the perturbation vanishes.

In this paper, we introduce a more general approach which consists in finding a particular part of the solution set of a given fixed point problem, i.e., fixed points which solve a variational inequality. More precisely, the goal of this paper is to present a method for finding hierarchically a fixed point of a nonexpansive semigroup $S = \{T(s)\}_{s \geq 0}$ with respect to another monotone operator A, namely,

Find $x^* \in Fix(S)$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(S). \tag{1.1}$$

This is an interesting topic due to the fact that it is closely related to convex programming problems. For the related works, refer to [1-19].

This paper is devoted to solve the problem (1.1). For this purpose, we propose a double-net algorithm which generates a net $\{x_{s,t}\}$ and prove that the net $\{x_{s,t}\}$ hierarchically converges to the solution of the problem (1.1), that is, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm as $s \to 0$ to a common fixed point $x_t \in Fix(S)$ of the nonexpansive semigroup $\{T(s)\}_{s \geq 0}$ and, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution x^* of the problem (1.1).

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $||\cdot||$, respectively. Recall a mapping $f : H \to H$ is called a contraction if there exists $\rho \in [0, 1)$ such that

$$||f(x) - f(y)|| \leq \rho ||x - y||, \quad \forall x, y \in H.$$

A mapping $T : C \to C$ is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in H.$$

Denote the set of fixed points of the mapping T by $Fix(T)$.

Recall also that a family $S := \{T(s)\}_{s \geq 0}$ of mappings of H into itself is called a nonexpansive semigroup if it satisfies the following conditions:

(S1) $T(0)x = x$ for all $x \in H$;
(S2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(S3) $||T(s)x - T(s)y|| \leq ||x - y||$ for all $x, y \in H$ and $s \geq 0$;
(S4) for all $x \in H$, $s \to T(s)x$ is continuous.

We denote by $Fix(T(s))$ the set of fixed points of $T(s)$ and by $Fix(S)$ the set of all common fixed points of S, i.e., $Fix(S) = \cap_{s \geq 0} Fix(T(s))$. It is known that $Fix(S)$ is closed and convex ([20], Lemma 1).

A mapping A of H into itself is said to be monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad \forall u, v \in H,$$

and $A : C \to H$ is said to be α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha ||Au - Av||^2, \quad \forall u, v \in H.$$

It is obvious that any α-inverse strongly monotone mapping A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Now, we introduce some lemmas for our main results in this paper.

Lemma 2.1. [21]*Let H be a real Hilbert space. Let the mapping* $A : H \to H$ *be* α-*inverse strongly monotone and* $\mu > 0$ *be a constant. Then, we have*

$$||(I - \mu A)x - (I - \mu A)y||^2 \leq ||x - y||^2 + \mu(\mu - 2\alpha)||Ax - Ay||^2, \quad \forall x, y \in H.$$

In particular, if $0 \leq \mu \leq 2\alpha$, *then* $I - \mu A$ *is nonexpansive.*

Lemma 2.2. [22]*Let C be a nonempty bounded closed convex subset of a Hilbert space H and* $\{T(s)\}_{s \geq 0}$ *be a nonexpansive semigroup on C. Then, for all* $h \geq 0$,

$$\lim_{t\to\infty}\sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)xds - T(h)\frac{1}{t}\int_0^t T(s)xds\right\| = 0.$$

Lemma 2.3. [23] (Demiclosedness Principle for Nonexpansive Mappings) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C converging weakly to a point $x \in C$ and $\{(I - T)x_n\}$ converges strongly to a point $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.4. *Let H be a real Hilbert space. Let $f : H \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$ and $A : H \to H$ be an α-inverse strongly monotone mapping. Let $\mu \in (0, 2\alpha)$ and $t \in (0, 1)$. Then, the variational inequality*

$$\begin{cases} x^* \in Fix(S); \\ \langle tf(z) + (1-t)(I-\mu A)z - z,\ x^* - z\rangle \geq 0, \quad \forall z \in Fix(S), \end{cases} \tag{2.1}$$

is equivalent to its dual variational inequality

$$\begin{cases} x^* \in Fix(S); \\ \langle tf(x^*) + (1-t)(I-\mu A)x^* - x^*,\ x^* - z\rangle \geq 0, \quad \forall z \in Fix(S). \end{cases} \tag{2.2}$$

Proof. Assume that $x^* \in Fix(S)$ solves the problem (2.1). For all $y \in Fix(S)$, set

$$x = x^* + s(y - x^*) \in Fix(S), \quad \forall s \in (0, 1).$$

We note that

$$\langle tf(x) + (1-t)(I-\mu A)x - x,\ x^* - x\rangle \geq 0.$$

Hence, we have

$$\langle tf(x^* + s(y-x^*)) + (1-t)(I-\mu A)(x^* + s(y-x^*)) - x^* - s(y-x^*),\ s(x^* - y)\rangle \geq 0,$$

which implies that

$$\langle tf(x^* + s(y-x^*)) + (1-t)(I-\mu A)(x^* + s(y-x^*)) - x^* - s(y-x^*),\ x^* - y\rangle \geq 0.$$

Letting $s \to 0$, we have

$$\langle tf(x^*) + (1-t)(I-\mu A)(x^*) - x^*,\ x^* - y\rangle \geq 0,$$

which implies the point $x^* \in Fix(S)$ is a solution of the problem (2.2).

Conversely, assume that the point $x^* \in Fix(S)$ solves the problem (2.2). Then, we have

$$\langle tf(x^*) + (1-t)(I-\mu A)x^* - x^*,\ x^* - z\rangle \geq 0.$$

Noting that $I - f$ and A are monotone, we have

$$\langle (I-f)z - (I-f)x^*,\ z - x^*\rangle \geq 0$$

and

$$\langle Az - Ax^*,\ z - x^*\rangle \geq 0.$$

Thus, it follows that

$$t\langle (I-f)z-(I-f)x^*,\ z-x^*\rangle + (1-t)\mu\langle Az-Ax^*,\ z-x^*\rangle \geq 0,$$

which implies that

$$\begin{aligned}\langle tf(z)+(1-t)(I-\mu A)z-z,\ x^*-z\rangle\\ &\geq \langle tf(x^*)+(1-t)(I-\mu A)x^*-x^*,\ x^*-z\rangle\\ &\geq 0.\end{aligned}$$

This implies that the point $x^* \in Fix(S)$ solves the problem (2.1). This completes the proof. □

3 Main results

In this section, we first introduce our double-net algorithm and then prove a strong convergence theorem for this algorithm.

Throughout, we assume that

(C1) H is a real Hilbert space;

(C2) $f: H \to H$ is a ρ-contraction with coefficient $\rho \in [0, 1)$, $A: H \to H$ is an α-inverse strongly monotone mapping, and $S = \{T(s)\}_{s\geq 0}: H \to H$ is a nonexpansive semigroup with $Fix(S) \neq \emptyset$;

(C3) the solution set Ω of the problem (1.1) is nonempty;

(C4) $\mu \in (0, 2\alpha)$ is a constant, and $\{\lambda_s\}_{0<s<1}$ is a continuous net of positive real numbers satisfying $\lim_{s\to 0}\lambda_s = +\infty$.

For any $s, t \in (0, 1)$, we define the following mapping

$$x \mapsto W_{s,t}x := s[tf(x) + (1-t)(x-\mu Ax)] + (1-s)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x d\nu.$$

We note that the mapping $W_{s,t}$ is a contraction. In fact, we have

$$\begin{aligned}\|W_{s,t}x - W_{s,t}y\| &= \Big\| s[tf(x)+(1-t)(x-\mu Ax)] + (1-s)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x d\nu\\ &\quad -s[tf(y)+(1-t)(y-\mu Ay)] - (1-s)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)y d\nu \Big\|\\ &\leq st\|f(x)-f(y)|| + s(1-t)||(x-\mu Ax)-(y-\mu Ay)||\\ &\quad +(1-s)||\frac{1}{\lambda_s}\int_0^{\lambda_s}(T(\nu)x - T(\nu)y)d\nu\Big\|\\ &\leq st\rho||x-y|| + s(1-t)||x-y|| + (1-s)||x-y||\\ &= [1-(1-\rho)st]||x-y||,\end{aligned}$$

which implies that $W_{s,t}$ is a contraction. Hence, by Banach's Contraction Principle, $W_{s,t}$ has a unique fixed point, which is denoted $x_{s,t} \in H$, that is, $x_{s,t}$ is the unique solution in H of the fixed point equation

$$\begin{aligned}x_{s,t} &= s[tf(x_{s,t}) + (1-t)(x_{s,t}-\mu Ax_{s,t})]\\ &\quad + (1-s)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu, \quad \forall s,\ t \in (0,1).\end{aligned} \tag{3.1}$$

Now, we give some lemmas for our main result.

Lemma 3.1. *For each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ defined by (3.1) is bounded.*

Proof. Taking any $z \in Fix(S)$, since $I - \mu A$ is nonexpansive (by Lemma 2.1), it follows from (3.1) that

$$\begin{aligned}
&\|x_{s,t} - z\| \\
= \; & \left\| s[tf(x_{s,t}) + (1-t)(I-\mu A)x_{s,t}] + (1-s)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu - z \right\| \\
\le \; & s\left\|tf(x_{s,t}) + (1-t)(I-\mu A)x_{s,t} - z\right\| + (1-s)\left\|\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu - z\right\| \\
\le \; & s\big[t\|f(x_{s,t}) - f(z)\| + t\|f(z)-z\| + (1-t)||(I-\mu A)x_{s,t} - (I-\mu A)z|| \\
& +(1-t)||(I-\mu A)z - z||\big] + (1-s)||x_{s,t} - z|| \\
\le \; & s[t\rho||x_{s,t} - z|| + t||f(z)-z|| + (1-t)||x_{s,t} - z|| + (1-t)\mu||Az||] \\
& +(1-s)||x_{s,t} - z|| \\
= \; & [1-(1-\rho)st]||x_{s,t} - z|| + st||f(z)-z|| + s(1-t)\mu||Az||.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{s,t} - z\| &\le \frac{1}{(1-\rho)t}(t||f(z)-z|| + (1-t)\mu||Az||) \\
&\le \frac{1}{(1-\rho)t}\max\{||f(z)-z||,\ \mu||Az||\}.
\end{aligned}$$

Thus, it follows that, for each fixed $t \in (0, 1)$, $\{x_{s,t}\}$ is bounded and so are the nets $\{f(x_{s,t})\}$ and $\{(I - \mu A)x_{s,t}\}$. This completes the proof. □

Lemma 3.2. *$x_{s,t} \to x_t \in Fix(S)$ as $s \to 0$.*

Proof. For each fixed $t \in (0, 1)$, we set $R_t := \frac{1}{(1-\rho)t}\max\{||f(z)-z||,\ \mu||Az||\}$. It is clear that, for each fixed $t \in (0, 1)$, $\{x_{s,t}\} \subset B(p, R_t)$, where $B(p, R_t)$ denotes a closed ball with the center p and radius R_t. Notice that

$$\left\|\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu - p\right\| \le ||x_{s,t} - p|| \le R_t.$$

Moreover, we observe that if $x \in B(p, R_t)$, then

$$||T(s)x - p|| \le ||T(s)x - T(s)p|| \le ||x - p|| \le R_t,$$

that is, $B(p, R_t)$ is $T(s)$-invariant for all $s \in (0, 1)$. From (3.1), it follows that

$$\begin{aligned}
\|T(\tau)x_{s,t} - x_{s,t}\| \le \; & \left\|T(\tau)x_{s,t} - T(\tau)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu\right\| \\
& + \left\|T(\tau)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu - \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu\right\| \\
& + \left\|\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)X_{s,t}\, d\nu - x_{s,t}\right\| \\
\le \; & \left\|T(\tau)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu - \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu\right\| \\
& +2\left\|x_{s,t} - \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)X_{s,t}\, d\nu\right\| \\
\le \; & 2s\left\|tf(x_{s,t}) + (1-t)(x_{s,t} - \mu Ax_{s,t}) - \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu\right\| \\
& + \left\|T(\tau)\frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu - \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)x_{s,t}d\nu\right\|.
\end{aligned}$$

By Lemma 2.2, for all $0 \le \tau < \infty$ and fixed $t \in (0, 1)$, we deduce

$$\lim_{s\to 0} \left\| T(\tau)x_{s,t} - x_{s,t} \right\| = 0. \tag{3.2}$$

Next, we show that, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ is relatively norm-compact as $s \to 0$. In fact, from Lemma 2.1, it follows that

$$||x_{s,t} - \mu A x_{s,t} - (z - \mu Az)||^2 \le ||x_{s,t} - z||^2 + \mu(\mu - 2\alpha)||Ax_{s,t} - Az||^2. \tag{3.3}$$

By (3.1), we have

$$\begin{aligned}
& ||x_{s,t} - z||^2 \\
= \;& st\langle f(x_{s,t}) - f(z),\ x_{s,t} - z\rangle + st\langle f(z) - z,\ x_{s,t} - z\rangle \\
& + s(1-t)\langle (I - \mu A)x_{s,t} - (I - \mu A)z,\ x_{s,t} - z\rangle \\
& + s(1-t)\langle (I - \mu A)z - z,\ x_{s,t} - z\rangle \\
& + (1-s)\left\langle \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)X_{s,t}\, d\nu\ - z,\ x_{s,t} - z\right\rangle \\
\le \;& st||f(x_{s,t}) - f(z)||\ ||x_{s,t} - z|| + st\langle f(z) - z,\ x_{s,t} - z\rangle \\
& + s(1-t)||(I - \mu A)x_{s,t} - (I - \mu A)z||\ ||x_{s,t} - z|| - s(1-t)\mu\langle Az,\ x_{s,t} - z\rangle \\
& + (1-s)\left\| \frac{1}{\lambda_s}\int_0^{\lambda_s} T(\nu)X_{s,t}\, d\nu\ - z||\ ||x_{s,t} - z \right\| \\
\le \;& st\rho||x_{s,t} - z||^2 + st\langle f(z) - z,\ x_{s,t} - z\rangle - s(1-t)\mu\langle Az,\ x_{s,t} - z\rangle \\
& + s(1-t)||(I - \mu A)x_{s,t} - (I - \mu A)z||\ ||x_{s,t} - z|| + (1-s)||x_{s,t} - z||^2 \\
\le \;& st\rho||x_{s,t} - z||^2 + st\langle f(z) - z,\ x_{s,t} - z\rangle - s(1-t)\mu\langle Az,\ x_{s,t} - z\rangle \\
& + \frac{s(1-t)}{2}(||(I - \mu A)x_{s,t} - (I - \mu A)z||^2 + ||x_{s,t} - z||^2) + (1-s)||x_{s,t} - z||^2.
\end{aligned}$$

This together with (3.3) imply that

$$\begin{aligned}
& ||x_{s,t} - z||^2 \\
\le \;& st\rho||x_{s,t} - z||^2 + st\langle f(z) - z,\ x_{s,t} - z\rangle - s(1-t)\mu\langle Az,\ x_{s,t} - z\rangle \\
& + \frac{s(1-t)}{2}(||x_{s,t} - z||^2 + \mu(\mu - 2\alpha)||Ax_{s,t} - Az||^2 + ||x_{s,t} - z||^2) + (1-s)||x_{s,t} - z||^2 \\
\le \;& [1 - (1-\rho)st]||x_{s,t} - z||^2 + st\langle f(z) - z,\ x_{s,t} - z\rangle \\
& - s(1-t)\mu\langle Az,\ x_{s,t} - z\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
& ||x_{s,t} - z||^2 \\
\le \;& \frac{1}{(1-\rho)t}\langle tf(z) + (1-t)(I - \mu A)z - z,\ x_{s,t} - z\rangle, \quad \forall z \in Fix(S).
\end{aligned} \tag{3.4}$$

Assume that $\{s_n\} \subset (0, 1)$ is such that $s_n \to 0$ as $n \to \infty$. By (3.4), we obtain immediately that

$$\begin{aligned}
& ||x_{s_n,t} - z||^2 \\
\le \;& \frac{1}{(1-\rho)t}\langle tf(z) + (1-t)(I - \mu A)z - z,\ x_{s_n,t} - z\rangle, \quad \forall z \in Fix(S).
\end{aligned} \tag{3.5}$$

Since $\{x_{s_n,t}\}$ is bounded, without loss of generality, we may assume that, as $s_n \to 0$, $\{x_{s_n,t}\}$ converges weakly to a point x_t. From (3.2) and Lemma 2.3, we get $x_t \in Fix(S)$.

Further, if we substitute x_t for z in (3.5), then it follows that

$$||x_{s_n,t} - x_t||^2 \leq \frac{1}{(1-\rho)t}\langle tf(x_t) + (1-t)(I-\mu A)x_t - x_t,\ x_{s_n,t} - x_t\rangle.$$

Therefore, the weak convergence of $\{x_{s_n,t}\}$ to x_t actually implies that $x_{s_n,t} \to x_t$ strongly. This has proved the relative norm-compactness of the net $\{x_{s,\ t}\}$ as $s \to 0$.

Now, if we take the limit as $n \to \infty$ in (3.5), we have

$$\begin{aligned}&\|x_t - z\|^2\\ &\leq \frac{1}{(1-\rho)t}\langle tf(z) + (1-t)(I-\mu A)z - z,\ x_t - z\rangle, \quad \forall z \in Fix(S).\end{aligned}$$

In particular, x_t solves the following variational inequality:

$$\begin{cases} x_t \in Fix(S); \\ \langle tf(z) + (1-t)(I-\mu A)z - z,\ x_t - z\rangle \geq 0, \quad \forall z \in Fix(S), \end{cases}$$

or the equivalent dual variational inequality (see Lemma 2.4):

$$\begin{cases} x_t \in Fix(S); \\ \langle tf(x_t) + (1-t)(I-\mu A)x_t - x_t,\ x_t - z\rangle \geq 0, \quad \forall z \in Fix(S). \end{cases} \tag{3.6}$$

Notice that (3.6) is equivalent to the fact that $x_t = P_{Fix(S)}(tf + (1\text{-}t)(I - \mu A))x_t$, that is, x_t is the unique element in $Fix(S)$ of the contraction $P_{Fix(S)}(tf + (1\text{-}t)(I - \mu A))$. Clearly, it is sufficient to conclude that the entire net $\{x_{s,\ t}\}$ converges in norm to $x_t \in Fix(S)$ as $s \to 0$. This completes the proof. □

Lemma 3.3. *The net* $\{x_t\}$ *is bounded.*

Proof. In (3.6), if we take any $y \in \Omega$, then we have

$$\langle tf(x_t) + (1-t)(I-\mu A)x_t - x_t,\ x_t - y\rangle \geq 0. \tag{3.7}$$

By virtue of the monotonicity of A and the fact that $y \in \Omega$, we have

$$\langle (I-\mu A)x_t - x_t,\ x_t - y\rangle \leq \langle (I-\mu A)y - y,\ x_t - y\rangle \leq 0. \tag{3.8}$$

Thus, it follows from (3.7) and (3.8) that

$$\langle f(x_t) - x_t,\ x_t - y\rangle \geq 0, \quad \forall y \in \Omega \tag{3.9}$$

and hence

$$\begin{aligned} \| x_t - y\|^2 &\leq \langle x_t - y,\ x_t - y\rangle + \langle f(x_t) - x_t,\ x_t - y\rangle \\ &= \langle f(x_t) - f(y),\ x_t - y\rangle + \langle f(y) - y,\ x_t - y\rangle \\ &\leq \rho \| x_t - y\|^2 + \langle f(y) - y,\ x_t - y\rangle. \end{aligned}$$

Therefore, we have

$$||x_t - y||^2 \leq \frac{1}{1-\rho}\langle f(y) - y,\ x_t - y\rangle, \quad \forall y \in \Omega. \tag{3.10}$$

In particular,

$$||x_t - y|| \leq \frac{1}{1-\rho}||f(y) - y||, \quad \forall t \in (0,1),$$

which implies that $\{x_t\}$ is bounded. This completes the proof. □

Lemma 3.4. *If the net $\{x_t\}$ converges in norm to a point $x^* \in \Omega$, then the point solves the variational inequality*

$$\langle (I-f)x^*,\ x-x^*\rangle \geq 0, \quad \forall x \in \Omega. \tag{3.11}$$

Proof. First, we note that the solution of the variational inequality (3.11) is unique.

Next, we prove that $\omega_w(x_t) \subset \Omega$, that is, if (t_n) is a null sequence in (0, 1) such that $x_{t_n} \to x'$ weakly as $n \to \infty$, then $x' \in \Omega$. To see this, we use (3.6) to get

$$\langle \mu A x_t,\ z-x_t\rangle \geq \frac{t}{1-t}\langle (I-f)x_t,\ x_t-z\rangle, \quad \forall z \in Fix(S).$$

However, since A is monotone, we have

$$\langle Az,\ z-x_t\rangle \geq \langle Ax_t,\ z-x_t\rangle.$$

Combining the last two relations yields that

$$\langle \mu Az,\ z-x_t\rangle \geq \frac{t}{1-t}\langle (I-f)x_t,\ x_t-z\rangle, \quad \forall z \in Fix(S). \tag{3.12}$$

Letting $t = t_n \to 0$ as $n \to \infty$ in (3.12), we get

$$\langle Az,\ z-x'\rangle \geq 0, \quad \forall z \in Fix(S),$$

which is equivalent to its dual variational inequality

$$\langle Ax',\ z-x'\rangle \geq 0, \quad \forall z \in Fix(S).$$

That is, x' is a solution of the problem (1.1) and hence $x' \in \Omega$.

Finally, we prove that $x' = x^*$, the unique solution of the variational inequality (3.11). In fact, by (3.10), we have

$$||x_{t_n}-x'||^2 \leq \frac{1}{1-\rho}\langle f(x')-x',\ x_{t_n}-x'\rangle, \quad \forall x' \in \Omega.$$

Therefore, the weak convergence to x' of $\{x_{t_n}\}$ implies that $x_{t_n} \to x'$ in norm. Thus, if we let $t = t_n \to 0$ in (3.10), then we have

$$\langle f(x')-x',\ y-x'\rangle \leq 0, \quad \forall y \in \Omega,$$

which implies that $x' \in \Omega$ solves the problem (3.11). By the uniqueness of the solution, we have $x' = x^*$ and it is sufficient to guarantee that $x_t \to x^*$ in norm as $t \to 0$. This completes the proof. □

Thus, by the above lemmas, we can obtain immediately the following theorem.

Theorem 3.5. *For each $(s, t) \in (0, 1) \times (0, 1)$, let $\{x_{s,\ t}\}$ be a double-net algorithm defined implicitly by (3.1). Then, the net $\{x_{s,\ t}\}$ hierarchically converges to the unique solution x^* of the hierarchical fixed point problem and the variational inequality problem (1.1), that is, for each fixed $t \in (0, 1)$, the net $\{x_{s,\ t}\}$ converges in norm as $s \to 0$ to a common fixed point $x_t \in Fix(S)$ of the nonexpansive semigroup $\{T(s)\}_{s \geq 0}$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution $x^* \in \Omega$ and the point x^**

also solves the following variational inequality.

$$\begin{cases} x^* \in \Omega; \\ \langle (I-f)x^*,\ x-x^* \rangle \geq 0, \quad \forall x \in \Omega. \end{cases}$$

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Author details
[1]Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, People's Republic of China [2]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea [3]Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan

Authors' contributions
All authors read and approved the final manuscript.

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